Rayliegh Distribution

Let 'X' +ve be a continuous random variable with interval (0,∞) is said to be Rayliegh distribution, having its p.d.f:

$$f(x) = 2\lambda x e^{-\lambda x^2} \qquad 0 \le x \le \infty$$

$$f(x) = 2\alpha x e^{-\lambda\alpha^2}$$

&

$$f(x) = \frac{2x}{\lambda} e^{\frac{-x^2}{\lambda}}$$

It has one parameter λ.

Properties:

i) Rayliegh distribution is a continuous distribution.

ii) The total area under the curve is unity.

iii) The range of the distribution is 0 to ∞.

iv) It has one parameter λ.

v) The mean of the Rayliegh distribution is $E(x) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$.

vi) The variance of the Rayliegh distribution is $Var(x) = \frac{4-\pi}{4\lambda}$.

Prove that total area under the curve is unity.

Proof:

Let by definition

Area= $\int f(x)dx$

As x ≈ Rayleigh (λ)

$$f(x) = 2\lambda x e^{-\lambda x^2} \qquad 0 \le x \le \infty$$

$$\text{Area} = \int_0^\infty 2\lambda x e^{-\lambda x^2} dx = 2\lambda \int_0^\infty x e^{-\lambda x^2} dx$$

$$\text{Area} = 2\lambda \int_0^\infty x e^{-\lambda x^2} dx \qquad \text{(i)}$$

Put $t = \lambda x^2 \Rightarrow \frac{t}{\lambda} = x^2 \Rightarrow x = \sqrt{\frac{t}{\lambda}} = \left(\frac{t}{\lambda}\right)^{1/2}$

$dx = \frac{1}{2}\left(\frac{t}{\lambda}\right)^{1/2-1} \frac{1}{\lambda} dt$ Put in (i)

Total area= $2\lambda \int_0^\infty \sqrt{\frac{t}{\lambda}} e^{-t} \frac{1}{2}\left(\frac{t}{\lambda}\right)^{\frac{1}{2}-1} \frac{1}{\lambda} dt$

Area= $\int_0^\infty \left(\frac{t}{\lambda}\right)^{\frac{1}{2}} e^{-t} \left(\frac{t}{\lambda}\right)^{-\frac{1}{2}} dt$

Area= $\int_0^\infty e^{-t} dt$

Area= $\int_0^\infty t^{1-1} e^{-t} dt$

By gamma function

Area=1

Find mean & variance of Rayleigh distribution

Solution: Let by definition

$$E(x)=\int xf(x)dx$$

As x ≈ Rayleigh (λ)

$$f(x)=2\lambda xe^{-\lambda x^2} \qquad 0\le x\le\infty$$

$$E(x)=\int_0^\infty x2\lambda xe^{\lambda x^2}dx$$

$$E(x)=2\lambda\int_0^\infty x^2e^{\lambda x^2}dx \qquad \text{(i)}$$

Put $t=\lambda x^2 \Rightarrow \frac{t}{\lambda}=x^2 \Rightarrow x=\sqrt{\frac{t}{\lambda}}=\left(\frac{t}{\lambda}\right)^{1/2}$

$dx=\frac{1}{2}\left(\frac{t}{\lambda}\right)^{1/2-1}\frac{1}{\lambda}dt$ Put in (i)

$$E(x)=2\lambda\int_0^\infty \frac{t}{\lambda}e^{-t}\frac{1}{2}\left(\frac{t}{\lambda}\right)^{\frac{1}{2}-1}\frac{1}{\lambda}dt=\int_0^\infty e^{-t}\left(\frac{t}{\lambda}\right)^{\frac{1}{2}+1-1}dt$$

$$E(x)=\int_0^\infty \frac{t^{\frac{1}{2}+1-1}}{\lambda^{\frac{1}{2}+1-1}}e^{-t}dt=\frac{1}{\lambda^{\frac{1}{2}}}\int_0^\infty t^{\frac{3}{2}-1}e^{-t}dt \qquad \text{(ii)}$$

As we know gamma function is

$$\Gamma a\, b^a=\int_0^\infty x^{a-1}e^{-x/b}dx \qquad \text{(iii)}$$

Comparing (ii) & (iii) then we get

$$a=\frac{3}{2},b=1$$

$$\Gamma a\, b^a=\Gamma\frac{3}{2}$$

Now (ii) becomes

$$E(x)=\frac{1}{\lambda^{\frac{1}{2}}}\Gamma\frac{3}{2}=\frac{1}{\sqrt{\lambda}}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)=\frac{1}{\sqrt{\lambda}}\left(\frac{1}{2}\right)\Gamma\frac{1}{2} \qquad \text{Therefore } \Gamma\frac{1}{2}=\sqrt{\pi}$$

$$E(x)=\frac{1}{\sqrt{\lambda}}\left(\frac{1}{2}\right)\sqrt{\pi}=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\lambda}}$$

Find variance of rayleigh distribution

Solution:

$$\operatorname{var}(x)=E(x^2)-[E(x)]^2$$

$$E(x^2)=\int x^2f(x)dx$$

As x ≈ Rayleigh (λ)

$$f(x)=2\lambda xe^{-\lambda x^2} \qquad 0\le x\le\infty$$

$$E(x^2)=\int_0^\infty x^2\,2\lambda xe^{-\lambda x^2}dx$$

$$E(x^2) = 2\lambda\int_0^\infty x^3 e^{-\lambda x^2} dx \qquad \text{(i)}$$

Put $t = \lambda x^2 \Rightarrow \frac{t}{\lambda} = x^2 \Rightarrow x = \sqrt{\frac{t}{\lambda}} = \left(\frac{t}{\lambda}\right)^{1/2}$

$dx = \frac{1}{2}\left(\frac{t}{\lambda}\right)^{1/2-1}\frac{1}{\lambda}dt$ Put in (i)

$$E(x^2) = 2\lambda\int_0^\infty \left(\sqrt{\frac{t}{\lambda}}\right)^3 e^{-t}\frac{1}{2}\left(\frac{t}{\lambda}\right)^{\frac{1}{2}-1}\frac{1}{\lambda}dt = \int_0^\infty \frac{t^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}}e^{-t}\frac{t^{\frac{1}{2}-1}}{\lambda^{\frac{1}{2}-1}}dt = \int_0^\infty \frac{t^{\frac{3}{2}+\frac{1}{2}-1}}{\lambda^{\frac{3}{2}+\frac{1}{2}-1}}e^{-t}dt = \int_0^\infty \frac{t^{\frac{3+1}{2}-1}}{\lambda}e^{-t}dt$$

$$E(x^2) = \frac{1}{\lambda}\int_0^\infty t^{\frac{3+1}{2}-1}e^{-t}dt \qquad \text{(ii)}$$

As we know gamma function is

$$\Gamma a\, b^a = \int_0^\infty x^{a-1}e^{-x/b}dx \qquad \text{(iii)}$$

Comparing (ii) & (iii) then we get

$$a = \frac{3+1}{2}, b = 1$$

$$\Gamma a\, b^a = \Gamma\frac{3+1}{2}$$

Now (A) becomes:

$$E(x^2) = \frac{1}{\lambda}\Gamma\frac{3+1}{2} = \frac{1}{\lambda}\left(\frac{3+1}{2}-1\right)\Gamma\left(\frac{3+1}{2}-1\right)$$

$$E(x^2) = \frac{1}{\lambda}\Gamma\frac{3+1}{2} = \frac{1}{\lambda}\left(\frac{2}{2}\right)\Gamma\frac{2}{2} = \frac{1}{\lambda}$$

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

$$\text{var}(x) = \frac{1}{\lambda} - \left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\right)^2 = \frac{1}{\lambda} - \frac{1}{4}\frac{\pi}{\lambda}$$

$$\text{var}(x) = \frac{4-\pi}{4\lambda}$$ Required result

Find the r^{th} moments about origin and use it to find mean & variance

Solution: Let by definition

$$\mu_r' = E(x^r) = \int_{-\infty}^{+\infty} x^r f(x)dx$$

As x ≈ Rayleigh (λ)

$$f(x) = 2\lambda x e^{-\lambda x^2} \qquad 0 \le x \le \infty$$

$$\mu_r' = \int_0^\infty x^r 2\lambda x e^{-\lambda x^2}dx = 2\lambda\int_0^\infty x^r x e^{-\lambda x^2}dx$$

$$\mu_r' = 2\lambda\int_0^\infty x^{r+1}e^{-\lambda x^2}dx \qquad \text{(i)}$$

Put $t=\lambda x^2 \Rightarrow \frac{t}{\lambda}=x^2 \Rightarrow x=\sqrt{\frac{t}{\lambda}}=\left(\frac{t}{\lambda}\right)^{1/2}$

$dx=\frac{1}{2}\left(\frac{t}{\lambda}\right)^{1/2-1}\frac{1}{\lambda}dt$ put in (i)

$$\mu_r'=2\lambda\int_0^\infty\left(\sqrt{\frac{t}{\lambda}}\right)^{r+1}e^{-t}\frac{1}{2}\left(\frac{t}{\lambda}\right)^{\frac{1}{2}-1}\frac{1}{\lambda}dt=\int_0^\infty\frac{t^{\frac{r+1}{2}}}{\lambda^{\frac{r+1}{2}}}e^{-t}\left(\frac{t}{\lambda}\right)^{\frac{1}{2}-1}dt=\int_0^\infty\frac{t^{\frac{r+1}{2}+\frac{1}{2}-1}}{\lambda^{\frac{r+1}{2}+\frac{1}{2}-1}}e^{-t}dt$$

$$\mu_r'=\frac{1}{\lambda^{\frac{r}{2}}}\int_0^\infty t^{\left(\frac{r}{2}+1\right)-1}e^{-t}dt \qquad \text{(ii)}$$

As we know that gamma function is

$$\Gamma(a)\,b^a=\int_0^\infty x^{a-1}e^{-x/b}dx \qquad \text{(iii)}$$

Comparing (ii) & (iii) then we get

a $=\frac{r}{2}+1$ & b =1

$$\Gamma(a)\,b^a=\Gamma\left(\frac{r}{2}+1\right)$$

Put in (ii)

$$\mu_r'=\frac{1}{\lambda^{\frac{r}{2}}}\Gamma\left(\frac{r}{2}+1\right) \qquad \text{(iv)}$$

Put r=1in (iv)

$$\mu_1'=\frac{1}{\lambda^{\frac{1}{2}}}\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{\lambda^{\frac{1}{2}}}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\frac{\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\lambda^{\frac{1}{2}}}=\frac{\sqrt{\pi}}{2\sqrt{\lambda}}=\frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \qquad :-\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

Put r = 2 in (iv)

$$\mu_2'=\frac{1}{\lambda^{\frac{2}{2}}}\Gamma\left(\frac{2}{2}+1\right)=\frac{1}{\lambda}$$

$$Var(x)=\mu_2'-\left(\mu_1'\right)^2$$

$$Var(x)=\mu_2=\frac{1}{\lambda}-\left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\right)^2=\frac{1}{\lambda}-\frac{\pi}{4\lambda}=\frac{4-\pi}{4\lambda}$$

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Derive the distribution function of Rayliegh distribution

Solution:

$$F(x) = P(X \le x) = f(X \le x)$$

$$F(X) = \int_{-\infty}^{x} f(x)dx$$

$$F(X) = \int_{0}^{x} 2\lambda x e^{-\lambda x^2} dx$$

$$F(X) = -\int_{0}^{x} (-2\lambda x) e^{-\lambda x^2} dx = -\int_{0}^{x} e^{-\lambda x^2} dx$$

$$F(X) = -\left[e^{-\lambda x^2}\right]_0^x = -\left[e^{-\lambda x^2} - e^{-0}\right]$$

$$F(X) = -\left[e^{-\lambda x^2} - 1\right] = -e^{-\lambda x^2} + 1 = 1 - e^{-\lambda x^2}$$ Required result

Find median of Rayliegh distribution

Solution: Let by definition of median:

$$P(X \le m) = \frac{1}{2}$$

$$\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$$

$$f(x) = 2\lambda x e^{-\lambda x^2}$$

$$\int_{0}^{m} 2\lambda x e^{-\lambda x^2} dx = \frac{1}{2}$$

$$\int_{0}^{\mu} (2\lambda x) e^{-\lambda x^2} dx = \frac{1}{2}$$

$$-\int_{0}^{x} (-2\lambda x) e^{-\lambda x^2} dx = \frac{1}{2}$$

$$-\int_{0}^{x} e^{-\lambda x^2} dx = \frac{1}{2}$$

$$-\left[e^{-\lambda x^2}\right]_0^\mu = \frac{1}{2}$$

$$-\left[e^{-\lambda \mu^2} - e^{-0}\right] = \frac{1}{2}$$

$$-\left[e^{-\lambda \mu^2} - 1\right] = \frac{1}{2}$$

$$-e^{-\lambda \mu^2} + 1 = \frac{1}{2}$$

$$-e^{-\lambda \mu^2} = \frac{1}{2} - 1$$

$$-e^{-\lambda \mu^2} = \frac{-1}{2}$$

$$e^{-\lambda \mu^2} = \frac{1}{2}$$

Taking log on both sides

$$-\lambda \mu^2 = \log(1/2) = log(1) - \log(2)$$

$$-\lambda \mu^2 = -\log(2)$$

$$\lambda \mu^2 = \log(2)$$

$$\mu^2 = \frac{\log 2}{\lambda} \qquad \mu = \sqrt{\frac{\log 2}{\lambda}}$$ Required result!

Find the mode of Rayliegh distribution

Solution: If following two conditions are satisfied then mode exists.

$f(x')=0$ or $\frac{d}{dx}\log f(x)=0$

$f(x')<0$ or $\frac{d^2}{dx^2}\log f(x)<0$

$f(x)=2\lambda x e^{-\lambda x^2}$ $\qquad 0\le x\le\infty$

Taking log on both sides

$$\log f(x)=\log\left(2\lambda x e^{-\lambda x^2}\right)=\log(2\lambda)+\log x-\lambda x^2\log e$$

$$\log f(x)=\log(2\lambda)+\log x-\lambda x^2$$

Differentiate w.r.t.to "x"

$$\frac{d\log f(x)}{dx}=0+\frac{1}{x}-2\lambda x\Rightarrow\frac{1}{x}-2\lambda x \qquad \text{(i)}$$

$$\frac{1}{x}=2\lambda x\Rightarrow\frac{1}{2\lambda}=x^2$$

$$x=\sqrt{\frac{1}{2\lambda}}$$

Again differentiate eq(i) w.r.t to 'x'

$$\frac{d^2\log f(x)}{dx^2}=-\frac{1}{x^2}-2\lambda=-\frac{1}{\frac{1}{2\lambda}}-2\lambda=-2\lambda-2\lambda=-4\lambda<0$$

Hence, both conditions are satisfied. So mode is:

$$\hat{x}=\sqrt{\frac{1}{2\lambda}}$$

Derive the Moment Generating Function m.g.f of Rayliegh distribution by following

P.d.f: $f(x)=2\lambda x e^{-\lambda x^2}$

Solution: Let by definition of m.g.f

$$M_0(t)=m_0(t)=E(e^{tx})$$

$$M_0(t)=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$M_0(t)=\int_0^{\infty}e^{tx}2\lambda x e^{-\lambda x^2}dx$$

$$M_0(t)=2\lambda\int_0^{\infty}e^{tx}xe^{-\lambda x^2}dx$$

$$M_0(t)=2\lambda\int_0^{\infty}xe^{-\lambda x^2+tx}dx$$

$$M_0(t)=2\lambda\int_0^{\infty}xe^{-\lambda\left(x^2-\frac{tx}{\lambda}\right)}dx$$

$$M_0(t)=2\lambda\int_0^{\infty}xe^{-\lambda\left(x^2-\frac{tx}{\lambda}+\left(\frac{t}{2\lambda}\right)^2-\left(\frac{t}{2\lambda}\right)^2\right)}dx$$

$$M_0(t) = 2\lambda \int_0^\infty x e^{-\lambda\left(x^2 - \frac{tx}{\lambda} + \left(\frac{t}{2\lambda}\right)^2 - \left(\frac{t}{2\lambda}\right)^2\right)} dx$$

$$M_0(t) = 2\lambda \int_0^\infty x e^{-\lambda\left(x - \frac{t}{\lambda}\right)^2 + \lambda\left(\frac{t}{2\lambda}\right)^2} dx$$

$$M_0(t) = 2\lambda \int_0^\infty x e^{-\lambda\left(x - \frac{t}{\lambda}\right)^2} e^{\lambda\left(\frac{t}{2\lambda}\right)^2} dx$$

$$M_0(t) = 2\lambda e^{\lambda\left(\frac{t}{2\lambda}\right)^2} \int_0^\infty x e^{-\lambda\left(x - \frac{t}{\lambda}\right)^2} dx$$

$$M_0(t) = 2\lambda e^{\lambda\left(\frac{t}{2\lambda}\right)^2} \int_0^\infty x e^{-\left(\frac{x - \frac{t}{\lambda}}{1/\sqrt{\lambda}}\right)^2} dx$$

$$M_0(t) = 2\lambda e^{\frac{t^2}{4\lambda}} \int_0^\infty x e^{-\left(\frac{x - \frac{t}{\lambda}}{1/\sqrt{\lambda}}\right)^2} dx$$

Put $y = \dfrac{x - \frac{t}{2\lambda}}{1/\sqrt{\lambda}}, \quad y\dfrac{1}{\sqrt{\lambda}} = x - \dfrac{t}{2\lambda}, \qquad x = y\dfrac{1}{\sqrt{\lambda}} + \dfrac{t}{2\lambda}$

$$dx = \frac{1}{\sqrt{\lambda}} dy$$

And the limits will be

As $x \to 0$ then $y \to -\dfrac{t}{2\sqrt{\lambda}}$

As $x \to \infty$ then $y \to \infty$

Now,

$$M_0(t) = 2\lambda e^{\frac{t^2}{4\lambda}} \int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} \left(y.\frac{1}{\sqrt{\lambda}} + \frac{t}{2\lambda}\right) e^{-y^2} \frac{1}{\sqrt{\lambda}} dy$$

$$M_0(t) = \frac{2\lambda e^{\frac{t^2}{4\lambda}}}{\sqrt{\lambda}} \left[\frac{1}{\sqrt{\lambda}} \int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} y e^{-y^2} dy + \frac{t}{2\lambda} \int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} e^{-y^2} dy\right]$$

$$= 2\sqrt{\lambda} e^{\left(t/2\sqrt{\lambda}\right)^2} \left[\frac{1}{-2\sqrt{\lambda}} \int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} e^{-y^2}(-2y) dy + \frac{t}{2\lambda} \int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} e^{-y^2} dy\right]$$

$$= 2\sqrt{\lambda} e^{\left(t/2\sqrt{\lambda}\right)^2} \left[\frac{-1}{2\sqrt{\lambda}}.e^{-y^2}\Big|_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} + \frac{t}{2\lambda} \int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty} e^{-y^2} dy\right]$$

$$M_0(t)=2\sqrt{\lambda}e^{\left(t/2\sqrt{\lambda}\right)^2}\left[\frac{-1}{2\sqrt{\lambda}}\left(-e^{-\left(\frac{t}{2\sqrt{\lambda}}\right)^2}\right)+\frac{t}{2\lambda}\int_{-\frac{t}{2\sqrt{\lambda}}}^{\infty}e^{-y^2}dy\right]$$

Put $\theta=\dfrac{t}{2\sqrt{\lambda}}$

$$M_0(t)=2\sqrt{\lambda}e^{\theta^2}\left[\frac{1}{2\sqrt{\lambda}}e^{-\theta^2}+\frac{\theta}{\sqrt{\lambda}}\int_{-\theta}^{\infty}e^{-y^2}dy\right]$$

By Gaussian Integration

$$H(\theta)=\frac{1}{\sqrt{\pi}}\int_{-\theta}^{\infty}e^{-y^2}dy$$

$$H(\theta)\sqrt{\pi}=\int_{-\theta}^{\infty}e^{-y^2}dy$$

Now

$$M_0(t)=2\sqrt{\lambda}e^{\theta^2}\left[\frac{1}{2\sqrt{\lambda}}e^{-\theta^2}+\frac{\theta}{\sqrt{\lambda}}H(\theta)\sqrt{\pi}\right]$$

$$M_0(t)=2\sqrt{\lambda}e^{\theta^2}\frac{1}{2\sqrt{\lambda}}e^{-\theta^2}+2\sqrt{\lambda}e^{\theta^2}\frac{\theta}{\sqrt{\lambda}}H(\theta)\sqrt{\pi}$$

$$M_0(t)=1+2e^{\theta^2}H(\theta)\sqrt{\pi}$$ That is the required result

Derive the Moment Generating Function m.g.f of Rayleigh distribution by following

P.d.f: $f(x)=\dfrac{2x}{\lambda^2}e^{-x^2/\lambda^2}$ $\quad 0\le x\le\infty$

Solution:

By definition of m.g.f:

$$M_0(t)=m_0(t)=E(e^{tx})$$

$$M_0(t)=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$M_0(t)=\int_{0}^{\infty}e^{tx}\frac{2x}{\lambda^2}e^{-x^2/\lambda^2}dx$$

$$M_0(t)=\frac{2}{\lambda^2}\int_{0}^{\infty}xe^{tx-\frac{x^2}{\lambda^2}}dx$$

$$M_0(t)=\frac{2}{\lambda^2}\int_{0}^{\infty}xe^{tx-\left(x^2/\lambda^2\right)}dx$$

$$M_0(t)=\frac{2}{\lambda^2}\int_{0}^{\infty}xe^{-\frac{1}{\lambda^2}\left(x^2-\lambda^2tx\right)}dx$$

$$M_0(t)=\frac{2}{\lambda^2}\int_{0}^{\infty}xe^{-\frac{1}{\lambda^2}\left(x^2-\lambda^2tx+\left(\frac{\lambda^2t}{2}\right)^2-\left(\frac{\lambda^2t}{2}\right)^2\right)}dx$$

$$M_0(t)=\frac{2}{\lambda^2}\int_0^\infty xe^{-\frac{1}{\lambda^2}\left(x-\frac{\lambda^2 t}{2}\right)^2+\frac{1}{\lambda^2}\left(\frac{\lambda^2 t}{2}\right)^2}dx$$

$$M_0(t)=\frac{2}{\lambda^2}e^{\frac{\lambda^4 t^2}{\lambda^2 4}}\int_0^\infty xe^{-\frac{1}{\lambda^2}\left(x-\frac{\lambda^2 t}{2}\right)^2}dx$$

$$M_0(t)=\frac{2}{\lambda^2}e^{\frac{\lambda^2 t^2}{4}}\int_0^\infty xe^{-\frac{1}{\lambda^2}\left(x-\frac{\lambda^2 t}{2}\right)^2}dx$$

$$M_0(t)=\frac{2}{\lambda^2}e^{\frac{\lambda^2 t^2}{4}}\int_0^\infty xe^{-\left(\frac{x-\frac{\lambda^2 t}{2}}{\sqrt{\lambda^2}}\right)^2}dx \qquad \text{(i)}$$

Put $Y=\left(\frac{x-\frac{\lambda^2 t}{2}}{\sqrt{\lambda^2}}\right)^2=\frac{x-\frac{\lambda^2 t}{2}}{\lambda}$

$\lambda y=x-\frac{\lambda^2 t}{2}$, $\qquad \lambda y+\frac{\lambda^2 t}{2}=x$

$dx=\lambda dy$

While limits will be:

When x →0 then y $\to -\frac{\lambda t}{2}$

When x →∞ then y →∞

$$M_0(t)=\frac{2}{\lambda^2}e^{\frac{\lambda^2 t^2}{4}}\int_{-\frac{\lambda t}{2}}^{\infty}\lambda\left(y+\frac{\lambda t}{2}\right)e^{-y^2}\lambda dy$$

$$M_0(t)=2e^{\frac{\lambda^2 t^2}{4}}\int_{-\frac{\lambda t}{2}}^{\infty}\left(y+\frac{\lambda t}{2}\right)e^{-y^2}dy$$

$$M_0(t)=2e^{\frac{\lambda^2 t^2}{4}}\left[\int_{-\frac{\lambda t}{2}}^{\infty}ye^{-y^2}dy+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\frac{\lambda^2 t^2}{4}}\left[-\frac{1}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}(-2y)dy+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\frac{\lambda^2 t^2}{4}}\left[-\frac{1}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}dy+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\frac{\lambda^2 t^2}{4}}\left[-\frac{1}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}dy+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty}e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\left(\frac{\lambda t}{2}\right)^2}\left[-\frac{1}{2}e^{-y^2}\Big|_{-\frac{\lambda t}{2}}^{\infty}+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty} e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\left(\frac{\lambda t}{2}\right)^2}\left[-\frac{1}{2}\left(e^{-\infty^2}-e^{-\left(\frac{-\lambda t}{2}\right)^2}\right)+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty} e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\left(\frac{\lambda t}{2}\right)^2}\left[(-\frac{1}{2})(-e^{-\left(\frac{\lambda t}{2}\right)^2})+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty} e^{-y^2}dy\right]$$

$$M_0(t)=2e^{\left(\frac{\lambda t}{2}\right)^2}\left[\frac{1}{2}e^{-\left(\frac{\lambda t}{2}\right)^2}+\frac{\lambda t}{2}\int_{-\frac{\lambda t}{2}}^{\infty} e^{-y^2}dy\right]$$

Put $\theta=\frac{\lambda t}{2}$

$$M_0(t)=2e^{(\theta)^2}\left[\frac{1}{2}e^{-(\theta)^2}+\theta\int_{-\theta}^{\infty} e^{-y^2}dy\right]$$

By Gaussian Integration

$$H(\theta)=\frac{1}{\sqrt{\pi}}\int_{-\theta}^{\infty}e^{-y^2}dy$$

$$H(\theta)\sqrt{\pi}=\int_{-\theta}^{\infty}e^{-y^2}dy$$

$$M_0(t)=2e^{(\theta)^2}\left[\frac{1}{2}e^{-(\theta)^2}+\theta H(\theta)\sqrt{\pi}\right]$$

$$M_0(t)=2e^{(\theta)^2}\frac{1}{2}e^{-(\theta)^2}+2e^{(\theta)^2}\theta.H(\theta)\sqrt{\pi}$$

$$M_0(t)=e^{(\theta)^2}e^{-(\theta)^2}+2e^{(\theta)^2}\theta.H(\theta)\sqrt{\pi}$$

$$M_0(t)=1+2e^{(\theta)^2}\theta.H(\theta)\sqrt{\pi}$$

Hence required m.g.f of Rayliegh distribution

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